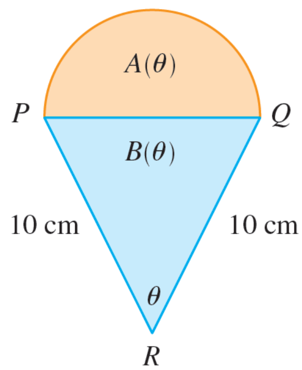
# Exercise 56

A semicircle with diameter $PQ$ sits on an isosceles triangle $PQR$ to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$




---

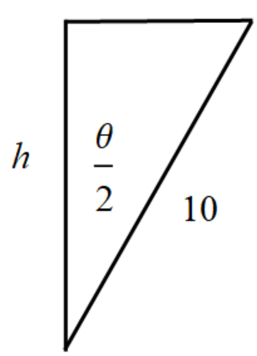
**Solution**

Use the law of cosines to find the diameter $d$ of the semicircle.

$$\begin{aligned} d^2 &= 10^2 + 10^2 - 2(10)(10)\cos\theta \\ &= 200 - 200\cos\theta \\ &= 200(1 - \cos\theta) \end{aligned}$$

Solve for $d$.

$$d = \sqrt{200(1 - \cos\theta)}$$




Determine the height of the isosceles triangle with trigonometry.

$$\cos\frac{\theta}{2} = \frac{h}{10}$$

$$h = 10\cos\frac{\theta}{2}$$

The two areas, $A(\theta)$ and $B(\theta)$, can now be found.

$$A(\theta) = \frac{1}{2}\pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{8}[200(1-\cos\theta)] = 25\pi(1-\cos\theta)$$

$$B(\theta) = \frac{1}{2}dh = \frac{1}{2}\left[\sqrt{200(1-\cos\theta)}\right]\left(10\cos\frac{\theta}{2}\right) = 5\sqrt{400}\sqrt{\frac{1-\cos\theta}{2}}\cos\frac{\theta}{2} = 100\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 50\sin\theta$$

Finally, calculate the desired limit by writing it in terms of other known limits.

$$\begin{aligned}
\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{25\pi(1-\cos\theta)}{50\sin\theta} &= \frac{\pi}{2}\lim_{\theta\to 0^+}\frac{1-\cos\theta}{\theta}\cdot\frac{\theta}{\sin\theta} \\
&= \frac{\pi}{2}\left(\lim_{\theta\to 0^+}\frac{1-\cos\theta}{\theta}\right)\left(\lim_{\theta\to 0^+}\frac{\theta}{\sin\theta}\right) \\
&= \frac{\pi}{2}\left(\lim_{\theta\to 0^+}\frac{1-\cos\theta}{\theta}\right)\left(\lim_{\theta\to 0^+}\frac{1}{\frac{\sin\theta}{\theta}}\right) \\
&= \frac{\pi}{2}\left(\lim_{\theta\to 0^+}\frac{1-\cos\theta}{\theta}\right)\left(\frac{1}{\lim\limits_{\theta\to 0^+}\frac{\sin\theta}{\theta}}\right) \\
&= \frac{\pi}{2}(0)\left[\frac{1}{(1)}\right] \\
&= 0
\end{aligned}$$